The subelliptic heat kernel on $\mathbf{SL}(2,\mathbb{R})$: an integral representation and some functional inequalities

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Abstract

In this paper, we study a subelliptic heat kernel on the Lie group $\mathbf{SL}(2,\mathbb{R})$. The subelliptic structure on $\mathbf{SL}(2,\mathbb{R})$ comes from the fibration $SO(2) \to \mathbf{SL}(2,\mathbb{R}) \to H^2$. First, we derive an integral representation for this heat kernel. This expression allows us to obtain some asymptotics in small times of this heat kernel and gives a way to compute the subriemannian distance. Then, we establish some gradient estimates and some functional inequalities like a Li-Yau type estimate and a reverse Poincaré inequality.

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1 Introduction

The goal of this work is to study a particular subelliptic structure on the Lie group $\mathbf{SL}(2,\mathbb{R})$. The structure we study is coming from the fibration: $SO(2) \to \mathbf{SL}(2,\mathbb{R}) \to H^2$ where H^2 is the 2-dimensional hyperbolic space. In this fibration the metric on $\mathbf{SL}(2,\mathbb{R})$ is the one inherited from the Killing form and is Lorentzian of signature (2,1). The restriction of this metric to the horizontal distribution is of signature (2,0) and gives the subellipitic structure (see [24] for more details). This space $\mathbf{SL}(2,\mathbb{R})$ can be proposed as the model space of a negatively curved 3-dimensional subriemannian manifold. To be more precise, it should be proposed as the model space of a 3-dimensional CR-manifold with vanishing pseudo-Hermitean torsion (Sasaki manifolds) and with constant negative curvature (see [11] for an account on CR-manifolds). This work is coming after some analogous studies on the Heisenberg group and on the canonical subelliptic Lie group $\mathbf{SU}(2)$. The subelliptic structure on this last group is very similar to the one studied here. The Heisenberg group plays the role of the Euclidean space in this geometry whereas $\mathbf{SU}(2)$ stands for the positively curved model space. As we will see it in the sequel, these three structures share a lot of results in common.

The precursor work before the study of the Heisenberg group is due to by Lévy who studied the area swept out by a two dimensionnal Brownian motion [21]. After that, the study of the heat kernel on the Heisenberg group began really with Hulanicki [17] and Gaveau [13]. In [13], Gaveau established an integral representation of the heat kernel which is now known as the Gaveau formula. This enabled him to obtain some asymptotics in small time of the heat kernel and even, more recently, with Beals and Greiner, to obtain some optimal bounds for the heat kernel (see [9] and also [22]). Recently, the focus was on obtaining functional inequalities and gradient estimates on this group. For example, a subcommutation between the gradient and the semi-group were derived in [12], [22] and [3].

In [8], a study of the subelliptic heat kernel on $\mathbf{SU}(2)$ was done. Our study here is very closed to this one since the structures are very similar. Using the isomorphism between $\mathbf{SU}(2)$ and the 3-sphere S^3, the autors managed to obtain an integral representation of the heat kernel on $\mathbf{SU}(2)$. This representation is based on the relations between the sublaplacian and the classical Laplace-Beltrami operator on S^3. Therefore the integral representation makes appear the classical heat kernel on S^3. Here on $\mathbf{SL}(2,\mathbb{R})$, even if $\mathbf{SL}(2,\mathbb{R})$ and the 3-dimensional hyperbolic space H^3 are not isomorphic, it is still possible to obtain an integral representation of the heat kernel in which the classical heat kernel on H^3 appears. In fact, here on $\mathbf{SL}(2,\mathbb{R})$ there is a relation between the sublaplacian and the Casimir operator (see remark 3.4), but we do not really use it to prove the integral representation of the heat kernel.

With this formula we are able to obtain some asymptotics in small time of the heat kernel, some ultracontractive bounds, a way to compute the subriemannian distance and the convergence of this diffusion towards the one on the Heisenberg group. We are also able to derive some gradient estimates and some functional inequalities like a Li-Yau type estimate and a reverse Poincaré inequality and some isoperimetric inequalities. These inequalities are also valid on $\mathbb{H}$ and $\mathbf{SU}(2)$ (see [3], [5] and [8]).

This paper is divided in three parts. In the first one, we recall some basics facts about the Lie

group $\mathbf{SL}(2,\mathbb{R})$ and express the coordinates we will use in the sequel. In the second one, we derive the integral representation of the heat kernel and give its consequences. In the last one, we establish some gradient estimates and some functional inequalities for the heat kernel.

2   Preliminaries on $\mathbf{SL}(2,\mathbb{R})$

The Lie group $\mathbf{SL}(2)$ is the group of 2×2, real matrices of determinant 1. Its Lie algebra $\mathfrak{sl}(2)$ consists of 2×2 matrices of trace 0. A basis of $\mathfrak{sl}(2)$ is formed by the matrices:

$$X = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \ Y = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \ Z = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),$$

for which the following relations hold

$$[X,Y] = 2Z, \quad [X,Z] = 2Y, \quad [Y,Z] = -2X. \tag{2.1}$$

We associate to these matrices the left-invariant vector fields they generate, which we still denote by the same letters. For example, for a smooth function f on $\mathbf{SL}(2,\mathbb{R})$ and $g \in \mathbf{SL}(2,\mathbb{R})$:

$$X(f)(g) = \lim_{t \to 0} \frac{1}{t} \left(f(g.\exp(tX)) - f(g) \right).$$

Below we will see the expressions of these vector fields in some coordinates.
Now we consider on this Lie group the left-invariant, second order differential operator

$$L = X^2 + Y^2$$

as well as the heat semigroup

$$P_t = e^{tL}.$$

Due to Hörmander's theorem and the structure of the Lie algebra (2.1), the operator L is subelliptic. Therefore the heat semi-group $(P_t)_{t>0}$ admits a smooth density with respect to its invariant measure.
The operator is subelliptic but not elliptic so that the associated geometry is not Riemannian but only subriemannian. The notion of distance associated to the operator L is given by

$$\delta(g_1, g_2) = \sup_{f \in \mathcal{C}} \{ |\ f(g_1) - f(g_2)\ | \}$$

where $\mathcal{C}$ is the set of smooth maps $\mathbf{SL}(2,\mathbb{R}) \to \mathbb{R}$ that satisfy $(Xf)^2 + (Yf)^2 \leq 1$. Via Chow's theorem, this distance can also be defined as the minimal length of horizontal curves joining two given points (see Chapter 3 of [7]). This distance is called the Carnot-Carathéodory distance. To study L, we introduce the cylindric coordinates:

$$(r, \theta, z) \to \exp\left(r\cos\theta X + r\sin\theta Y \right)\exp(zZ)$$

$$= \left(\begin{array}{cc} \cosh(r)\cos(z) + \sinh(r)\cos(\theta + z) & \cosh(r)\sin(z) + \sinh(r)\sin(\theta + z) \\ -\cosh(r)\sin(z) + \sinh(r)\sin(\theta + z) & \cosh(r)\cos(z) - \sinh(r)\cos(\theta + z) \end{array} \right),$$

with
$$r > 0, \ \theta \in [0, 2\pi], \ z \in [-\pi, \pi].$$

These coordinates are the equivalent in our context of the ones which were used in [8] to study a similar subellpitic operator on the Lie group $\mathbf{SU}(2)$.

Simple but tedious computations show that in these coordinates, the left-regular representation sends the matrices X, Y and Z to the left-invariant vector fields:

$$X = \cos(\theta + 2z)\frac{\partial}{\partial r} - \sin(\theta + 2z)\left(\tanh r\frac{\partial}{\partial z} + \left(\frac{1}{\tanh r} - \tanh r\right)\frac{\partial}{\partial \theta}\right),$$

$$Y = \sin(\theta + 2z)\frac{\partial}{\partial r} + \cos(\theta + 2z)\left(\tanh r\frac{\partial}{\partial z} + \left(\frac{1}{\tanh r} - \tanh r\right)\frac{\partial}{\partial \theta}\right),$$

$$Z = \frac{\partial}{\partial z}.$$

We therefore obtain

$$L = X^2 + Y^2$$
$$= \frac{\partial^2}{\partial r^2} + 2\coth 2r\frac{\partial}{\partial r} + \tanh^2 r\frac{\partial^2}{\partial z^2} + \frac{4}{\sinh^2 2r}\frac{\partial^2}{\partial \theta^2} + (1 - \tanh^2 r)\frac{\partial^2}{\partial \theta \partial z}.$$

The invariant and, in fact, also symmetric measure for L is then given (up to a constant) by

$$d\mu = \frac{\sinh 2r}{2}dr d\theta dz.$$

The choice of the constant is made to obtain a good convergence towards the Lebesgue measure of $\mathbb{R}^3$ which is the invariant measure for the Heisenberg group (see section 3.3). Recall the group $\mathbf{SL}(2, \mathbb{R})$ is unimodular and note that the invariant measure μ coincides with the bi-invariant Haar measure of the group. Note also that L commutes with $\frac{\partial}{\partial \theta}$ and with $\frac{\partial}{\partial z}$. From the commutation with $\frac{\partial}{\partial \theta}$, we deduce that the heat kernel (issued from the identity) only depends on (r, z). It will then be denoted by $p_t(r, z)$.

3 The subelliptic heat kernel

3.1 Integral representation of the kernel

Let us consider the second order differential operator on the interval $[1, \infty)$

$$\mathcal{J} = (x^2 - 1)\frac{d^2}{dx^2} + 3x\frac{d}{dx}$$

with invariant and symmetric measure $(x^2 - 1)^{1/2}$. It is well known (see [26]) that the heat kernel s_t associated to $\mathcal{J}$ issued from 1 has the following expression for $x \geq 1$:

$$s_t(x) = \frac{e^{-t}}{(4\pi t)^{3/2}}\left(\frac{\mathrm{arch}x}{\sqrt{x^2 - 1}}\right)e^{-\frac{(\mathrm{arch}x)^2}{4t}}. \tag{3.2}$$

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That is, for f a smooth function $[1, \infty) \to \mathbb{R}$,

$$(e^{t\mathcal{J}} f)(1) = \int_1^\infty s_t(x) f(x)(x^2 - 1)^{1/2} dx.$$

It is clear the function $x \to (\text{arch} x)^2$ admits an holomorphic extension to $\mathbb{C} - \{]\infty, 1]\}$; but in fact, using Schwarz symmetry principle, we can see that this extension is holomorphic on $\mathbb{C} - \{]\infty, -1]\}$. Therefore this is the same for its derivative: $x \to \frac{\text{arch} x}{\sqrt{x^2 - 1}}$. So the heat kernel s_t itself admits an holomorphic extension to $\mathbb{C} - \{]\infty, -1]\}$. By setting $x = \cosh r, r \geq 0$, we have

$$s_t(\cosh r) = \frac{e^{-t}}{(4\pi t)^{3/2}} \left(\frac{r}{\sinh r} \right) e^{-\frac{r^2}{4t}}. \tag{3.3}$$

This heat kernel corresponds in fact to the one on the 3-dimensional hyperbolic space.
Now easy calculations give us that s_t satisfies the following expressions:

$$\partial_t s_t(\cosh r \cos z) = \Delta_1(s_t(\cosh r \cos z)) \tag{3.4}$$

where $\Delta_1 = \partial_{r,r}^2 + 2 \coth 2r \partial_r + (\tanh^2 r - 1)\partial_{z,z}^2$ and

$$\partial_t s_t(\cosh r \cosh y) = \Delta_2(s_t(\cosh r \cosh y)) \tag{3.5}$$

where $\Delta_2 = \partial_{r,r}^2 + 2 \coth 2r \partial_r + (1 - \tanh^2 r)\partial_{y,y}^2$.
Δ_1 and Δ_2 are two self-adjoint operators respectively on $(0, \infty) \times [-\pi, \pi]$ and on $(0, \infty) \times (0, \infty)$ with respective symmetric measure $\frac{\sinh 2r}{2} dr dz$ and $\frac{\sinh 2r}{2} dr dy$. Δ_1 is a hyperbolic operator whereas Δ_2 is an elliptic operator. For a geometric interpretation of Δ_1, see remark 3.4.

Lemma 3.1 *If f is a smooth function $(0, \infty) \times (0, \infty) \to \mathbb{R}$, then for $t \geq 0$,*

$$(e^{t\Delta_2} f)(0) = \frac{1}{2} \int_{r>0} \int_{y>0} s_t(\cosh r \cosh y) f(r, y) \sinh 2r dr dy$$

Proof. Indeed we saw that s_t satisfies the equation:

$$\partial_t s_t(\cosh r \cosh y) = \Delta_2(s_t(\cosh r \cosh y)).$$

Now we must check the initial condition and show that for a smooth function $f: (0, \infty) \times (0, \infty) \to \mathbb{R}$:

$$\frac{1}{2} \int_{r>0} \int_{y>0} s_t(\cosh r \cosh y) f(r, y) \sinh 2r dr dy$$

Since we will make the following change of variables:

$$\begin{cases} u &= \cosh r \cosh y \\ v &= \cosh r \sinh y \end{cases}$$

we take the function f of the form $f(r, z) = g(\cosh r \cosh z)h(\cosh r \sinh z)$. The new domain is $D = \{(u, v), u \geq 1, v \geq 0, u^2 - v^2 \geq 1\}$ and the Jacobian determinant is $\frac{1}{2} \sinh 2r$. So

$$\int_{r>0} \int_{y>0} s_t(\cosh r \cosh y) g(\cosh r \cosh y) h(\cosh r \sinh y) \frac{\sinh 2r}{2} dr dy$$

$$= \int \int_D s_t(u) g(u) h(v) du dv$$

$$= \int_{u \geq 1} \left(\int_0^{(1-u^2)^{1/2}} h(v) dv \right) s_t(u) g(u) du$$

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We may rewrite it as

$$\int_{u \geq 1} s_t(u) l(u) (u^2 - 1)^{1/2} du$$

where l is the continuous fonction

$$l(u) = g(u) \left(\frac{\int_0^{(u^2-1)^{1/2}} h(v) dv}{(u^2 - 1)^{1/2}} \right).$$

Now, since s_t is the heat kernel of a diffusion issued from 1 with respect to the measure $(u^2 - 1)^{1/2} du$ and l is continuous, the last quantity is converging towards $l(1) = g(1) h(0) = f(0,0)$ and the lemma is proved. $\qquad\square$

Remark 3.2 *The function f in the lemma 3.1 is defined on the space $(0, \infty) \times (0, \infty)$ and not on $(0, \infty) \times [-\pi, \pi]$ as a radial function (i.e. a function which does not depend on the variable θ) on $\mathbf{SL}(2, \mathbb{R})$ should be.*

Proposition 3.3 *We have for $t > 0, r > 0, z \in [-\pi, \pi]$,*

$$
\begin{aligned}
p_t(r, z) &= \frac{1}{2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} s_t(\cosh r \cosh y) dy \\
&= \frac{1}{2} \frac{e^{-t}}{(4\pi t)^2} \int_{-\infty}^{+\infty} e^{-\frac{\operatorname{arch}^2(\cosh r \cosh y) - (y-iz)^2}{4t}} \frac{\operatorname{arch}(\cosh r \cosh y)}{\sqrt{\cosh^2 r \cosh^2 y - 1}} dy
\end{aligned}
$$

Proof. The second equality is just obtained by using the explicit value of s_t and shows that the integral is well defined since it is absolutely convergent. Now let

$$q_t(r, z) = \frac{1}{2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} s_t(\cosh r \cosh y) dy.$$

By using the fact that

$$\frac{\partial}{\partial t} \left(\frac{e^{\frac{(y-iz)^2}{4t}}}{\sqrt{4\pi t}} \right) = \frac{\partial^2}{\partial z^2} \left(\frac{e^{\frac{(y-iz)^2}{4t}}}{\sqrt{4\pi t}} \right) = -\frac{\partial^2}{\partial y^2} \left(\frac{e^{\frac{(y-iz)^2}{4t}}}{\sqrt{4\pi t}} \right)$$

and

$$\frac{\partial}{\partial t}(s_t(\cosh r \cosh y)) = \left(\partial_{r,r}^2 + 2 \coth 2r \partial_r + (1 - \tanh^2 r) \partial_{y,y}^2 \right) (s_t(\cosh r \cosh y)),$$

a double integration by parts with respect to the variable y shows that

$$\frac{\partial q_t}{\partial t} = \frac{1}{2} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{\frac{(y-iz)^2}{4t}} \Delta_3(s_t(\cosh r \cosh y)) dy$$

where $\Delta_3 = \partial_{r,r}^2 + 2 \coth 2r \partial_r - \tanh^2 r \partial_{y,y}^2$.

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Now another double integration by parts in the variable y shows us that

$$\frac{\partial}{\partial t} q_t(r, z) = L q_t(r, z).$$

Let us now check the initial condition. Let $f(r, z) = e^{imz} g(r)$ where $m \in \mathbb{Z}$ and g is a smooth function. We have

$$\int_{r>0} \int_{z=-\pi}^{\pi} q_t(r, z) f(r, z) \frac{\sinh 2r}{2} dr dz$$

$$= \frac{1}{2} \int_{r>0} \int_{z=-\pi}^{\pi} \int_{y>0} \left(\frac{e^{-\frac{(z+iy)^2}{4t}} + e^{-\frac{(z-iy)^2}{4t}}}{\sqrt{4\pi t}} \right) s_t(\cosh r \cosh y) g(r) e^{imz} \frac{\sinh 2r}{2} dr dz dy$$

and by integrating with respect to the z variable along a rectangle in the complex plane, we get

$$\int_{z=-\pi}^{\pi} \left(\frac{e^{-\frac{(z+iy)^2}{4t}}}{\sqrt{4\pi t}} \right) e^{imz} dz$$

$$= e^{my} \int_{z=-\pi}^{\pi} \left(\frac{e^{-\frac{z^2}{4t}}}{\sqrt{4\pi t}} \right) e^{imz} dz + 2(-1)^m e^{-\frac{\pi^2}{4t}} \int_{u=0}^{y} \frac{e^{-\frac{(y-u)^2}{4t}}}{4\pi t} \sin\left(\frac{\pi(y-u)}{2t} \right) e^{mu} du$$

We can do the same for the other term and eventually obtain

$$\int_{r>0} \int_{z=-\pi}^{\pi} q_t(r, z) f(r, z) \sinh 2r \, dr dz$$

$$= \left(\int_{z=-\pi}^{\pi} \frac{e^{-\frac{z^2}{4t}}}{\sqrt{4\pi t}} e^{imz} dz \right) \int_{r>0} \int_{y>0} s_t(\cosh r \cosh y) g(r) \cosh(my) \frac{\sinh 2r}{2} dr dy$$

$$+ \quad 2(-1)^m e^{-\frac{\pi^2}{4t}} \int_{r>0} \int_{y>0} \int_{u=0}^{y} \frac{e^{-\frac{(y-u)^2}{4t}}}{\sqrt{4\pi t}} \sin\left(\frac{\pi(y-u)}{2t} \right) \sinh(mu) s_t(\cosh r \cosh y) g(r) \frac{\sinh 2r}{2} du dr dy$$

The term in the first line is equal to $a(t) e^{t\Delta_2}(l)(0)$ where l is the function $l(r, y) = g(r) \cosh(my)$ and $a(t) = \int_{z=-\pi}^{\pi} \frac{e^{-\frac{z^2}{4t}}}{\sqrt{4\pi t}} e^{imz} dz$. By classical results on the heat kernel on $\mathbb{R}$, $a(t)$ tends to $e^{im0} = 1$ when t goes to 0. Similarly $e^{t\Delta_2}(l)(0)$ tends to $l(0)$. Therefore this term is converging to $g(0) = f(0, 0)$ when t goes to 0. We can check the term on the second line is converging to 0 when t goes to 0 which gives us the desired convergence and ends our proof. $\qquad\square$

Remark 3.4 *It is not the way we used to prove it, but we have the following geometric interpretation for the sublaplacian:*

$$L = \Delta_1 + Z^2.$$

Δ_1 *is, in fact, the Casimir operator* $\Delta_1 = X^2 + Y^2 - Z^2$ *(see [25]). As* Δ_1 *is in the center of the envelopping algebra of* $\mathbf{SL}(2, \mathbb{R})$ *(and if fact generates it), we have also the following geometric interpretation:*

$$e^{tL} = e^{tZ^2} e^{t\Delta_1}.$$

3.2 Asymptotics of the heat kernel in small time

The goal of this section is to obtain the precise asymptotics of the heat kernel when $t \to 0$. We start with the points of the form $(0, z)$ that lie on the cut-locus of 0. We restrict ourselves to the points with $z > 0$. For these points we have

$$p_t(0, z) = \frac{1}{2} \frac{e^{-t}}{(4\pi t)^2} e^{-\frac{z^2}{4t}} \int_{-\infty}^{+\infty} e^{\frac{-iyz}{2t}} \frac{y}{\sinh y} dy.$$

A computation of the integral is possible using residus calculus and gives the following:

Proposition 3.5 *For $z \in (0, \pi]$ and $t > 0$,*

$$p_t(0, z) = \frac{e^{-t}}{16t^2} \frac{e^{-\frac{2\pi z + z^2}{4t}}}{\left(1 + e^{-\frac{\pi z}{2t}}\right)^2}$$

therefore, there exists a constant C such that when $t \to 0$, for $z \in (0, \pi]$

$$p_t(0, z) = \frac{e^{-t}}{16t^2} e^{-\frac{2\pi z + z^2}{4t}} \left(1 + O(e^{-\frac{C}{t}})\right)$$

By continuity of the heat kernel we obtain the value on the diagonal.

Proposition 3.6 *For $t > 0$,*

$$p_t(0, 0) = \frac{e^{-t}}{64t^2}.$$

Now we turn to points of the form $(r, 0)$ and give their asymptotics for the heat kernel.

Proposition 3.7 *For $r > 0$, when $t \to 0$,*

$$p_t(r, 0) \sim \frac{1}{2} \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{r}{\sinh r} \sqrt{\frac{1}{r \coth r - 1}} e^{-\frac{r^2}{4t}}.$$

Proof. We have for $r > 0$

$$p_t(r, 0) = \frac{1}{2} \frac{e^{-t}}{(4\pi t)^2} \int_{-\infty}^{+\infty} e^{-\frac{\operatorname{arch}^2(\cosh r \cosh y) - y^2}{4t}} \frac{\operatorname{arch}(\cosh r \cosh y)}{\sqrt{\cosh^2 r \cosh^2 y - 1}} dy$$

We now analyze the above integral in small times thanks to the Laplace method. On $\mathbb{R}$, the function

$$f(y) = \operatorname{arch}(\cosh r \cosh y))^2 - y^2$$

has a unique minimum which is attained at $y = 0$ and is equal to r^2, at this point:

$$f''(0) = 2(r \coth r - 1).$$

The result follows by the Laplace method. $\qquad\square$

The previous proposition can be extended by the same method when $z \neq 0$. Let $r > 0, z \in [-\pi, \pi]$ and consider the function

$$f(y) = (\operatorname{arch}(\cosh r \cosh y))^2 - (y - iz)^2,$$

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This function is well defined and holomorphic on the strip $|\mathrm{Im}(y)| < \mathrm{arcos}\left(\frac{-1}{\cosh r}\right)$ and it has a critical point at $i\theta(r,z)$ where $\theta(r,z)$ is the unique solution in $\left(-\mathrm{arcos}\left(\frac{-1}{\cosh r}\right), \mathrm{arcos}\left(\frac{-1}{\cosh r}\right)\right)$ to the equation:

$$\theta(r,z) - z = \cosh r \sin\theta(r,z) \frac{\mathrm{arch}(\cosh r \cos\theta(r,z))}{\sqrt{\cosh^2 r \cos^2\theta(r,z) - 1}}.$$

Indeed the function $\theta \rightarrow \cosh r \sin\theta(r,z)\frac{\mathrm{arch}(\cosh r \cos\theta(r,z))}{\sqrt{\cosh^2 r \cos^2\theta(r,z)-1}}$ is continuous, strictly increasing from $-\infty$ to ∞ and with a derivative greater than 1.

At the critical point, $f''(i\theta(r,z))$ is a positive and real number

$$f''(i\theta(r,z)) = 2\frac{\sinh^2 r}{u(r,z)^2 - 1}\left[\frac{u(r,z)\mathrm{arch}u(r,z)}{\sqrt{u(r,z)^2 - 1}} - 1\right]$$

with $u(r,z) = \cosh r \cos\theta(r,z)$ since $u > -1$.

We may observe that z and $\theta(r,z)$ have opposite signs.

By the same method than in the previous proposition, we obtain:

Proposition 3.8 *Let $r > 0, z \in [-\pi, \pi]$. When $t \rightarrow 0$,*

$$p_t(r,z) \sim \frac{1}{\sinh r}\frac{\mathrm{arccosh}u(r,z)}{\sqrt{\frac{u(r,z)\mathrm{arccosh}u(r,z)}{\sqrt{u^2(r,z)-1}} - 1}}\frac{\sqrt{\pi}e^{-\frac{(\theta(r,z)-z)^2\tanh^2 r}{4t\sin^2\theta(r,z)}}}{4t^{\frac{3}{2}}}$$

with $u(r,z) = \cosh r \cos\theta(r,z)$.

Remark 3.9 *According to Léandre results [18] and [19] (see also [16]), the previous asymptotics give a way to compute the subriemannian distance from 0 to the point $(r, \theta, z) \in \mathbf{SL}(2,\mathbb{R})$ by computing $\lim_{t\rightarrow 0} -4t \ln p_t(r,z)$. This distance does not depend on the variable θ and shall be denoted by $d(r,z)$.*

- *For $z \in [-\pi, \pi]$,*
$$d^2(0,z) = 2\pi \mid z \mid + z^2.$$

- *For $r > 0$,*
$$d^2(r,0) = r^2.$$

- *For $z \in [-\pi, \pi], r > 0$,*
$$d^2(r,z) = \frac{(\theta(r,z) - z)^2 \tanh^2 r}{\sin^2\theta(r,z)}.$$

From this remark we can get some estimates of the distance:

Proposition 3.10 *There exist two constants $c, C > 0$ such that for all $r > 0$ and $z \in [-\pi, \pi]$:*

$$c(r^2 + |z|) \leq d(r,z) \leq C(r^2 + |z|).$$

Proof. For the right inequality, as in our coordinates on the group $\mathbf{SL}(2,\mathbb{R})$, $(r,0,0) * (0,0,z) = (r,0,z)$, we obtain by using the left invariance of the distance: $d(r,z) \leq d(r,0) + d(0,z)$. By combining it with the previous result, for all $r > 0$ and $z \in [-\pi, \pi]$, we get:

$$d^2(r,z) \leq C(r^2 + |z|)$$

where C is a positive constant.

Let us turn to the left inequality. Since $(r,0,z) * (0,0,-z) = (r,0,0)$, then $d(r,0) - d(0,z) \leq d(r,z)$ and so the result is true in the region where $r^2 \geq A|z|$ with A big enough.
Similarly, since $(r,\pi,0) * (r,0,z) = (0,0,z)$ then $d(0,z) - d(r,0) \leq d(r,z)$ and the result is true in the region where $|z| \geq Br^2$ with B big enough. Now, consider the region $\{(r,z), \frac{1}{A}r^2 \leq |z| \leq Br^2\}$. Since $|z|$ is bounded by π, r is also bounded above on this region. Recall that $\theta(r,z)$ and z have opposite signs. Therefore

$$\frac{(\theta(r,z) - z)^2}{\sin^2 \theta(r,z)} \geq 1 + 2|z|.$$

But as r is bounded above, there exists a constant c' such that $\tanh^2 r \geq c'r^2$. So on this domain the expression of the distance gives:

$$c'r^2(1 + |z|) \leq d^2(r,z).$$

On this domain the function on the left side behaves like $r^2 + |z|$ and gives the result. $\qquad\square$

The proposition 3.6 gives that the heat kernel satisfies the following ultracontractivity bound:

$$p_t(0,0) = ||p_t||_\infty \leq \frac{e^{-t}}{64t^2}. \tag{3.6}$$

Now by using theorem 1.1 of Grigor'yan [15], this leads to the following gaussian upper estimate:

Proposition 3.11 *For all $\varepsilon > 0$, there exist two positive constants C_ε and δ_ε such that*

$$p_t(r,z) \leq C_\varepsilon \frac{e^{-\delta_\varepsilon t}}{t^2} \exp\left(-\frac{d^2(r,z)}{4(1+\varepsilon)t}\right).$$

Now, let us have a look to the measure of the subriemannian balls. Consider the Riemannian metric obtain by setting that (X, Y, Z) is an orthonormal frame of the tangent space in each point. Call δ_R the induced distance. By the very definition of the distances, it is clear that the subriemannian distance δ is greater than the Riemannian one δ_R, then $B(g,\rho) \subset B_R(g,\rho)$ where $B_R(g,\rho)$ is the Riemannian ball of center g and radius ρ and $B(g,\rho)$ the subriemannian one. Moreover, in our case the canonical Riemannian volume measure is proportional to the subriemannian invariant measure μ. Note that, as we are on a left-invariant Lie group, the Ricci tensor is the same in each point and therefore bounded from below by a constant $-K$ with $K > 0$.
Therefore, for all $g \in \mathbf{SL}(2,\mathbb{R})$ and all $\rho > 0$

$$\mu(B(g,\rho)) \leq \mu(B_R(g,\rho)) \leq C_1 \exp(C_2\rho)$$

for two positive constants C_1 and C_2.

3.3 From $\mathbb{SL}(2, \mathbb{R})$ to Heisenberg

Let us first recall some basic properties of the three-dimensional Heisenberg group (see by e.g. [7], [3] and the references therein): $\mathbb{H}$ can be represented as $\mathbb{R}^3$ endowed with the polynomial group law:

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2 - x_2 y_1).$$

The left invariant vector fields read in cylindric coordinates $(x = r \cos \theta, y = r \sin \theta)$:

$$\tilde{X} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} - r \sin \theta \frac{\partial}{\partial z} \tag{3.7}$$

$$\tilde{Y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} + r \cos \theta \frac{\partial}{\partial z} \tag{3.8}$$

$$\tilde{Z} = \frac{\partial}{\partial z}. \tag{3.9}$$

And the following equalities hold

$$[\tilde{X}, \tilde{Y}] = 2\tilde{Z}, \ [\tilde{X}, \tilde{Z}] = [\tilde{Y}, \tilde{Z}] = 0.$$

We denote

$$\tilde{L} = \tilde{X}^2 + \tilde{Y}^2$$

and

$$\tilde{\Gamma}(f, f) = (\tilde{X} f)^2 + (\tilde{Y} f)^2.$$

Due to Gaveau's formula (see [17], [13]), with respect to the Lebesgue measure $r dr d\theta dz$ the heat kernel associated to the semigroup $(\tilde{P}_t)_{t \geq 0} = (e^{t\tilde{L}})_{t \geq 0}$ writes

$$h_t(r, z) = \frac{1}{16\pi^2} \int_{-\infty}^{+\infty} e^{\frac{i\lambda z}{2}} \frac{\lambda}{\sinh \lambda t} e^{-\frac{r^2}{4} \lambda \coth \lambda t} d\lambda. \tag{3.10}$$

From a metric point of view it is known that the Heisenberg group is the tangent cone in the Gromov-Hausdorff sense. This means that balls of radius R for a dilating distance on $\mathbf{SL}(2, \mathbb{R})$ are getting closer and closer in a certain sense of the balls of the same radius R of the Heisenberg group. For a precise statement of it, see Mitchell theorem [23] (see also [7]). Here we will see some more precise results.

First, in our setting, the dilation of $\mathbf{SL}(2, \mathbb{R})$ towards the Heisenberg group can be seen at the level of differential operators.

Indeed through the map

$$\begin{aligned} \mathbf{SL}(2, \mathbb{R}) &\rightarrow \mathbb{H} \\ \exp(r(\cos \theta X + \sin \theta Y)) \exp zZ &\rightarrow (r, \theta, z) \end{aligned}$$

we can see the vector fields X, Y and Z of $\mathbf{SL}(2, \mathbb{R})$ as first order differential operators acting on smooth functions on the Heisenberg group whose supports are included in the box $[0, \infty) \times [0, 2\pi] \times [-\pi, \pi]$.

Let us now denote by D the dilation vector field on $\mathbb{H}$ given in cylindric coordinates by

$$D = r\frac{\partial}{\partial r} + 2z\frac{\partial}{\partial z}$$

For $c \geq 1$ we denote by X^c, Y^c and Z^c the dilated vector fields

$$X^c = \frac{1}{\sqrt{c}}e^{-\frac{1}{2}\ln cD}Xe^{\frac{1}{2}\ln cD},$$

$$Y^c = \frac{1}{\sqrt{c}}e^{-\frac{1}{2}\ln cD}Ye^{\frac{1}{2}\ln cD},$$

$$Z^c = \frac{1}{\sqrt{c}}e^{-\frac{1}{2}\ln cD}Ze^{\frac{1}{2}\ln cD}.$$

In the cylindric coordinates of the Heisenberg group, we have

$$X^c = \cos(\theta + \frac{2z}{c})\frac{\partial}{\partial r} - \sin(\theta + \frac{2z}{c})\left(\sqrt{c}\tanh\frac{r}{\sqrt{c}}\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{c}\tanh\frac{r}{\sqrt{c}}} - \frac{\tanh\frac{r}{\sqrt{c}}}{\sqrt{c}}\right)\frac{\partial}{\partial\theta}\right),$$

$$Y^c = \sin(\theta + \frac{2z}{c})\frac{\partial}{\partial r} + \cos(\theta + \frac{2z}{c})\left(\sqrt{c}\tanh\frac{r}{\sqrt{c}}\frac{\partial}{\partial z} + \left(\frac{1}{\sqrt{c}\tanh\frac{r}{\sqrt{c}}} - \frac{\tanh\frac{r}{\sqrt{c}}}{\sqrt{c}}\right)\frac{\partial}{\partial\theta}\right),$$

$$Z^c = \frac{\partial}{\partial z},$$

so that the dilated vector fields are well-defined on the box $[0,\infty) \times [0, 2\pi] \times [-\sqrt{c}\pi, \sqrt{c}\pi]$. Consequently, if $f : \mathbb{H} \to \mathbb{R}$ is a smooth function with compact support, we can speak of $X^c f$, $Y^c f$, and $Z^c f$ as soon as the dilation factor c is big enough. For the dilated sublaplacian

$$L^c = \frac{1}{c}e^{-\frac{1}{2}\ln cD}Le^{\frac{1}{2}\ln cD}$$
$$= (X^c)^2 + (Y^c)^2$$
$$= \frac{\partial^2}{\partial r^2} + \frac{2}{\sqrt{c}}\mathrm{cotanh}\frac{2r}{\sqrt{c}}\frac{\partial}{\partial r} + \frac{1}{c}\left(\frac{1}{\tanh\frac{r}{\sqrt{c}}} - \tanh\frac{r}{\sqrt{c}}\right)^2\frac{\partial^2}{\partial\theta^2} + c\tanh^2\frac{r}{\sqrt{c}}\frac{\partial^2}{\partial z^2} + 2(1 - \tanh^2\frac{2r}{\sqrt{c}})\frac{\partial^2}{\partial z\partial\theta},$$

the same remarks hold true.

With these notations, the *operator* analogue of the convergence of dilated $\mathbf{SL}(2,\mathbb{R})$ to $\mathbb{H}$ is the following:

Proposition 3.12 *If $f : \mathbb{H} \to \mathbb{R}$ is a smooth function with compact support, then, uniformly,* $\lim_{c\to+\infty} X^c f = \tilde{X}f$, $\lim_{c\to\infty} Y^c f = \tilde{Y}f$, $\lim_{c\to\infty} Z^c f = \tilde{Z}f$, $\lim_{c\to\infty} L^c f = \tilde{L}f$.

As a corollary, we obtain the following:

Corollary 3.13 *Uniformly on compact sets of* $\mathbb{R}_{\geq 0} \times \mathbb{R}$,

$$\lim_{t \to 0} \frac{d(\sqrt{t}r, tz)}{\sqrt{t}} = d_{\mathbb{H}}(r, z)$$

where $d_{\mathbb{H}}$ is the Carnot-Carathéodory distance of the point (r, θ, z) to the origin in $\mathbb{H}$.

Now we can prove the stronger result for the diffusions.

Proposition 3.14 *Uniformly on compact sets of* $\mathbb{R}_{\geq 0} \times \mathbb{R}$,

$$\lim_{t \to 0} t^2 p_t(\sqrt{t}r, tz) = h_1(r, z)$$

The computations to prove this result are based on the explicit formula for p_t and are very closed from the ones done in [8] on the group $\mathbf{SU}(2)$ for the same result, and therefore the proof will be omit.

4 Some functional inequalities for the heat kernel

In this section we will obtain some functional inequalities for the heat kernel and in particular some gradient bounds for the heat kernel. Let us recall that

$$L = X^2 + Y^2$$

with

$$[X, Y] = 2Z, \quad [X, Z] = 2Y, \quad [Y, Z] = -2X.$$

Since we will use it a lot in the sequel we introduce the following notations (see [1], [2]). For f and g smooth functions on $\mathbf{SL}(2, \mathbb{R})$, let

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$$

and

$$\Gamma_2(f, g) = \frac{1}{2}(L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)).$$

In the present setting, we obtain

$$\Gamma(f, f) = X^2 + Y^2$$

and

$$\Gamma_2(f, f) = (X^2 f)^2 + (Y^2 f)^2 + \frac{1}{2}((XY + YX)f)^2 + 2(Zf)^2 - 4\Gamma(f, f) - 4(Xf)(YZf) + 4(Yf)(XZf).$$

4.1 Γ_2 radial

In this section, we will express the Γ and the Γ_2 of a smooth radial function f (i.e. that only depends on the variables r and z).

$$\Gamma(f,f) = \left(\frac{\partial f}{\partial r}\right)^2 + \tanh^2 r \left(\frac{\partial f}{\partial z}\right)^2,$$

and

$$\Gamma_2(f,f) = \left(\frac{\partial^2 f}{\partial r^2}\right)^2 + \left(\frac{2}{\sinh 2r}\frac{\partial f}{\partial r} - \tanh^2 r \frac{\partial^2 f}{\partial z^2}\right)^2 + 2\left(\frac{1}{\cosh^2 r}\frac{\partial f}{\partial z} + \tanh r \frac{\partial^2 f}{\partial r \partial z}\right)^2.$$

Thus, we obtain that for a smooth radial function f, $\Gamma_2(f,f) \geq 0$. This is an interesting fact which may be surprising if we think that this subelliptic $\mathbf{SL}(2,\mathbb{R})$ is the subelliptic model space with negative curvature.

4.2 A first gradient bound

Proposition 4.1 *Let $f : \mathbf{SL}(2,\mathbb{R}) \to \mathbb{R}$ be a smooth function. For $t > 0$ and $g \in \mathbf{SL}(2,\mathbb{R})$,*

$$\Gamma(P_t f, P_t f)(g) \leq A(t) \left(\int_{\mathbf{SL}(2,\mathbb{R})} f^2 d\mu - \left(\int_{\mathbf{SL}(2,\mathbb{R})} f d\mu\right)^2\right)$$

where

$$A(t) = -\frac{1}{4}\frac{\partial}{\partial t}\int_{\mathbf{SL}(2,\mathbb{R})} p_t^2 d\mu.$$

Proof. The proof is exactly the same as the one on the $\mathbf{SU}(2)$ group and on the Heisenberg group (see [3] and [8]). $\square$

Remark 4.2 *Due to the use of the Cauchy-Schwarz inequality in the previous proof, we see that the previous inequality is sharp.*

We now study the constant $A(t)$.

Proposition 4.3 *We have the following properties:*

- *A is decreasing;*
- *$A(t) \sim_{t \to 0} \frac{1}{2^9 t^3}$;*
- *$A(t) \sim_{t \to +\infty} \frac{e^{-2t}}{2^9 t^2}$.*

Proof. Let us first show that A is decreasing. We have:

$$A'(t) = -\int_{\mathbf{SL}(2,\mathbb{R})} \Gamma_2(p_t, p_t) d\mu.$$

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Since p_t only depends on (r, z), $\Gamma_2(p_t, p_t) \geq 0$ and thus $A'(t) \leq 0$.

We can now observe that, due to the semigroup property,

$$\int_{\mathbf{SL}(2,\mathbb{R})} p_t^2 d\mu = p_{2t}(0)$$

and

$$p_t(0, 0) = \frac{e^{-t}}{(4\pi t)^2} \int_{-\infty}^{+\infty} \frac{y}{\sinh y} dy = \frac{e^{-t}}{64t^2}.$$

$\square$

4.3 Li-Yau type inequality

We now provide a Li-Yau type estimate for the heat semigroup. This inequality appears in [5] but all its consequences do not appear in this paper. The idea of its proof is close to the one done in [6] for elliptic operators. Let us recall it:

Proposition 4.4 *For all $\alpha > 2$, for every positive function f and every $t > 0$,*

$$\Gamma(\ln P_t f) + \frac{4t}{\alpha}(Z \ln P_t f)^2 \leq \left(\frac{3\alpha - 1}{\alpha - 1} + \frac{t}{2\alpha}\right)\frac{LP_t f}{P_t f} + \frac{16t}{\alpha} + \frac{4(3\alpha - 1)}{\alpha - 1} + \frac{(3\alpha - 1)^2}{4(\alpha - 2)}\frac{1}{t}. \quad (4.11)$$

Let us denote

$$A(t) = \frac{3\alpha - 1}{\alpha - 1} + \frac{t}{2\alpha}$$

and

$$B(t) = \frac{16t}{\alpha} + \frac{4(3\alpha - 1)}{\alpha - 1} + \frac{(3\alpha - 1)^2}{4(\alpha - 2)}\frac{1}{t}.$$

$A(t)$ and $B(t)$ here are always non negative. For t small, $A(t)$ is of the order of a constant and $B(t)$ is of order of $\frac{C}{t}$.
For, t big, one can choose $\alpha = t$ and get both $A(t)$ and $B(t)$ of the order of a constant.

Remark 4.5 *It can be shown that with this choice $\alpha = t$, the constants $A(t)$ and $B(t)$ are of the best order possible in the differential system that appears in [5].*

As a direct corollary of the Li-Yau type inequality of proposition 4.4, we classically deduce (by integrating along geodesics) the following Harnack type inequalities:

Proposition 4.6 *There exist two positive constant A_1 and A_2 such that for $0 < t_1 < t_2 < 1$ and $g_1, g_2 \in \mathbf{SL}(2, \mathbb{R})$*

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{A_1} \exp\left(A_2 \frac{\delta(g_1, g_2)^2}{t_2 - t_1}\right) \quad (4.12)$$

and there exist two positive constants $\tilde{A}_1$ and $\tilde{A}_2$ such that for $2 < t_1 < t_2$ and $g_1, g_2 \in \mathbf{SL}(2, \mathbb{R})$

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \exp\left(\tilde{A}_1(t_2 - t_1)\right) \exp\left(\tilde{A}_2 \frac{\delta(g_1, g_2)^2}{t_2 - t_1}\right) \quad (4.13)$$

where $\delta(g_1, g_2)$ denotes the Carnot-Caratheodory distance from g_1 to g_2.

As another corollary of the Li-Yau inequality, we can also prove the following global estimate:

Proposition 4.7 *There exists a constant $C > 0$ such that for $t \in (0,1)$, $r > 0$, $z \in [-\pi, \pi]$,*

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq C \left(\frac{d(r,z)}{t} + \frac{1}{\sqrt{t}} \right),$$

and there exists a constant $\tilde{C} > 0$ such that for $t > 2$, $r > 0$, $z \in [-\pi, \pi]$,

$$\sqrt{\Gamma(\ln p_t)(r,z)} \leq \tilde{C} \left(\frac{d(r,z)}{t} + 1 \right),$$

Proof. The proof is the same as on $\mathbf{SU}(2)$ (see [8]) since it is only based on the preceeding Harnack inequalities and the positivity of the Γ_2 of a radial function. The only difference is that in the second point we have to use the Harnack inequality (4.13) in big times. $\square$

4.4 The reverse spectral gap inequality

As in the Heisenberg group case and in the $\mathbf{SU}(2)$ case (see [3] and [8]), we can easily obtain a reverse Poincare inequality with a sharp constant for the subelliptic heat kernel measure on $\mathbf{SL}(2, \mathbb{R})$.

Proposition 4.8 *Let $f : \mathbf{SL}(2, \mathbb{R}) \to \mathbb{R}$ be a smooth function. For $t > 0$ and $g \in \mathbf{SL}(2, \mathbb{R})$,*

$$\Gamma(P_t f, P_t f)(g) \leq C(t) \left(P_t f^2(g) - (P_t f)^2(g) \right)$$

where

$$C(t) = -\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbf{SL}(2,\mathbb{R})} p_t \ln p_t d\mu.$$

Proof. As before, the proof is exactly the same as on Heisenberg and on the $\mathbf{SU}(2)$ group (see [3] and [8]). $\square$

Remark 4.9 *Due to the use of the Cauchy-Schwarz inequality in the previous proof, we see that the previous inequality is sharp.*

We now study the constant

$$C(t) = -\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbf{SL}(2,\mathbb{R})} p_t \ln p_t d\mu.$$

Let us recall:

$$\int_{\mathbf{SL}(2,\mathbb{R})} \frac{\Gamma(p_t, p_t)}{p_t} d\mu = \int_{\mathbf{SL}(2,\mathbb{R})} \Gamma(\ln p_t, \ln p_t) p_t d\mu = - \int_{\mathbf{SL}(2,\mathbb{R})} \ln p_t L p_t d\mu = -\frac{\partial}{\partial t} \int_{\mathbf{SL}(2,\mathbb{R})} p_t \ln p_t d\mu.$$

Proposition 4.10 *We have the following properties:*

- *C is decreasing;*

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- $C(t) \sim_{t \to 0} \frac{1}{t}$;

Proof. Let us first show that C is decreasing. After some computations, we obtain:

$$C'(t) = -\int_{\mathbf{SL}(2,\mathbb{R})} \Gamma_2(\ln p_t, \ln p_t) p_t d\mu.$$

But now, let us observe that p_t only depends on (r, z). Therefore $\Gamma_2(\ln p_t, \ln p_t) \geq 0$ and thus $C'(t) \leq 0$.

We now study $C(t)$ when $t \to 0$. The idea is that, asymptotically when $t \to 0$, the constant $C(t)$ has to behave like the best constant of the reverse spectral gap inequality on the Heisenberg group (see the Section 3.3). From [3], this constant is known to be $1/t$. This is also the case for $\mathbf{SU}(2)$ (see [8]). Let $0 < t < 1$ we have:

$$tC(t) = \frac{t}{2} \int_{\mathbf{SL}(2,\mathbb{R})} p_t \Gamma(\ln p_t, \ln p_t) d\mu$$

$$= \int_{r>0} \int_{z=-\frac{\pi}{t}}^{\frac{\pi}{t}} t^{5/2} \frac{\sinh 2\sqrt{t}r}{2} p_t(\sqrt{t}r, tz) \Gamma(\ln p_t, \ln p_t)(\sqrt{t}r, tz) dr dz$$

Now, by using the result of Section 3.3, we easily obtain that, uniformly on compact sets, the following convergences hold

$$\lim_{t \to 0} t^{3/2} \frac{\sinh 2\sqrt{t}r}{2} p_t(\sqrt{t}r, tz) = h_1(r, z)r$$

$$\lim_{t \to 0} t\Gamma(\ln p_t, \ln p_t)(\sqrt{t}r, tz) = \tilde{\Gamma}(\ln h_1)(r, z),$$

where $h_t(r, z)$ and $\tilde{\Gamma}$ are defined in Section 3.3 (see 3.10, 3.7, 3.8).

So we obtain the desired convergence on any compact subsets $K = [0, R] \times [-A, A]$, that is

$$\int_0^R \int_{-A}^A t^{5/2} \frac{\sinh 2\sqrt{t}r}{2} p_t(\sqrt{t}r, tz) \Gamma(\ln p_t, \ln p_t)(\sqrt{t}r, tz) dr dz \to_{t \to 0} \int_0^R \int_{-A}^A h_1(r, z) \tilde{\Gamma}(\ln h_1)(r, z) r dr dz.$$

Now we have also to control the integrand on the outside of the compact K. Thanks to Proposition 4.7, there exists a constant $C > 0$ such that

$$t\Gamma(\ln p_t, \ln p_t)(\sqrt{t}r, tz) \leq C \left(1 + \frac{d(\sqrt{t}r, tz)}{\sqrt{t}} \right)^2, \quad t \in (0, 1).$$

and thanks to proposition 3.11, there exist two constant $C_1, C_2 > 0$ such that

$$t^2 p_t(\sqrt{t}r, tz) \leq C_1 \exp \left(-C_2 \frac{d^2(\sqrt{t}r, tz)}{t} \right).$$

Also we have:

$$\frac{\sinh 2\sqrt{t}r}{\sqrt{t}} \leq e^{2r}.$$

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Eventually, the estimates of the distance of proposition 3.10 show that the integral outside the compact is going to 0. Therefore:

$$\lim_{t\to 0} tC(t) = \frac{1}{2}\int_{\mathbb{R}^3} h_1(r,z)\tilde{\Gamma}(\ln h_1)(r,z)rdrd\theta dz.$$

This last expression is equal to 1, according to [3].

$\square$

Remark 4.11 *We can ask about the behaviour of $C(t)$ as t goes to infinity. By using proposition 4.4 and its notation, for a positive function f,*

$$\int P_t(f)\Gamma(\ln P_t f)d\mu \le B(t)\int fd\mu.$$

By taking f an approximation of the unity, we obtain:

$$C(t) \le B(t).$$

And so for big t, $C(t)$ is less than a constant we can compute.

4.5 Some isoperimetrics inequalities on $\mathbf{SL}(2,\mathbb{R})$

We can now recover some isoperimetric results from the Li-Yau inequality. We use methods of Varopoulos and Ledoux (see [27] and [20]). First we set:

Proposition 4.12 *There exists C such that for every smooth function f on $\mathbf{SL}(2,\mathbb{R})$ and every $0 < t < 1$,*

$$||\sqrt{\Gamma P_t f}||_\infty \le \frac{C}{\sqrt{t}}||f||_\infty.$$

There exists C' such for every smooth function f on $\mathbf{SL}(2,\mathbb{R})$ and $0 < t < 1$,

$$||f - P_t f||_1 \le C'\sqrt{t}||\sqrt{\Gamma f}||_1.$$

Proof. Indeed, for the first point, the Li-Yau inequality gives for $0 < t < 1$ and f a positive function:

$$L(P_t f)^- \le \frac{C}{t}P_t f.$$

By integrating against μ and noticing $\int L(P_t f)d\mu = 0$, we get

$$\frac{1}{2}\int |L(P_t f)|d\mu \le \frac{C}{t}\int fd\mu.$$

Then $||LP_t f||_1 \le \frac{2C}{t}||f||_1$, and since LP_t is self-adjoint, by duality $||LP_t f||_\infty \le \frac{2C}{t}||f||_\infty$. By pluging-in this result in the Li-Yau inequality (4.4),

$$\Gamma P_t f \le \frac{C'}{t}||f||_\infty P_t f$$

which implies the first result.

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For the second point, let f and g be two smooth functions,

$$\int g(P_t f - f)d\mu = \int_0^t \int gLP_s f d\mu ds$$

$$= -\int_0^t \int \Gamma(P_s g, f)d\mu ds$$

Since $\Gamma(P_s g, f) \leq \sqrt{\Gamma P_s g}\sqrt{\Gamma f}$, by the first point, we have

$$|\int g(P_t f - f)d\mu| \leq C||g||_\infty \int_0^t \frac{1}{\sqrt{s}}ds \int \sqrt{\Gamma f}d\mu$$

$$= 2C\sqrt{t}||g||_\infty \int \sqrt{\Gamma f}d\mu.$$

By letting g tend to $sign(P_t f - f)$, we end the proof. $\square$

And actually these last results will enable us to obtain some isoperimetric inequalities on small sets. For A and B measurable sets, let us denote

$$K_t(A, B) = \int_B P_t(1_A)d\mu.$$

It is easy to see that

$$K_t(A, A^c) = \mu(A) - K_t(A, A)$$

and

$$K_t(A, A) = ||P_{\frac{t}{2}}1_A||_2^2.$$

We have the following proposition:

Proposition 4.13 *Let A be a measurable set of $\mathbf{SL}(2, \mathbb{R})$ which is a Cacciopoli set and call $P(A)$ its perimeter (see [14] and the references therein to see their definition in our context) then*

$$K_t(A, A^c) \leq 2C\sqrt{t}P(A). \tag{4.14}$$

Now assume also $\mu(A)$ is small enough, then

$$\mu(A)^{\frac{Q-1}{Q}} \leq CP(A)$$

for some positive constant C and $Q = 4$ stands for the homegenous dimension of the group.

Proof. Let A be a measurable set of $\mathbf{SL}(2, \mathbb{R})$ and let f and g be two smooth functions which aproximate respectively 1_A and 1_{A^c} and with $||g||_\infty \leq 1$. Then the quantity $\int g(P_t f - f)d\mu$ converges towards $K_t(A, A^c)$ and as before

$$\int g(P_t f - f)d\mu \leq ||g||_\infty ||P_t f - f||_1$$

$$\leq 2C\sqrt{t} \int \sqrt{\Gamma f}d\mu$$

As it is well known, we can choose f such that $\int \sqrt{\Gamma f}d\mu$ tends towards $P(A)$ (see theorem 1.14 of [14]), so we obtain

$$K_t(A, A^c) \leq 2C\sqrt{t}P(A).$$

Therefore,

$$P(A) \geq \frac{C'}{\sqrt{t}}(\mu(A) - ||P_{\frac{t}{2}}1_A||_2^2).$$

Using the ultracontractivity in small times, we get $||P_t f||_\infty \leq \frac{C}{t^{Q/2}}||f||_1$ and by interpolation $||P_t f||_2 \leq \frac{\sqrt{C}}{t^{Q/4}}||f||_1$, so

$$P(A) \geq \frac{C'}{\sqrt{t}}\mu(A)\left(1 - \frac{C}{\left(\frac{t}{2}\right)^{Q/2}}\mu(A)\right).$$

Now we will have to optimize the function of t on the right-hand side. We see this function attains a positive maximum for t of the order $\mu(A)^{\frac{2}{Q}}$ which has value of order $\mu(A)^{\frac{Q-1}{Q}}$. $\qquad\square$

Remark 4.14 *In all our previous results, we can give an explicit bound on the constants that appeared from the constant that appeared in the Li-Yau inequality 4.11.*

Remark 4.15 *It is known that the result of proposition 4.13 is true for all sets (see theorem 7.5 of [10] and note that the space* $\mathbf{SL}(2,\mathbb{R})$ *has constant curvature* $R = -1$*). It seems that the proposition 4.4 is far from being optimal in big times.*

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